ALL SHORTEST ROUTES IN A GRAPH

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A shortest route is sought between every pair of nodes (i, j) in a graph when directed arc distances a_{ij} are given, where the values of a_{ij} may be positive, negative, or zero except $a_{ii} = 0$. If the graph is incomplete so that an arc (i, j) is missing, the value of $a_{ij} = \infty$. This problem (as is well known) includes the travelling salesman problem since the route for (i,i) is a cycle and one can solve a travelling salesman problem with distances $d_{ij} > 0$ by finding a minimum cycle in a graph $[a_{ij} = d_{ij} - K]$ where $K > \Sigma_i \Sigma_j d_{ij}$. Our objective, therefore is more modest, it is to find a negative cycle in a graph if one exists, if none then to find all the shortest routes.

The procedure is inductive and was stimulated by a remark of Ralph Gomory's that an inductive approach was probably as efficient as any other. It is not certain, however, whether this procedure has appeared elsewhere in the literature and so is presented here. It is shown that $n(n-1)^2$ additions and an equal number of comparisons are required to solve an n node problem. This number can be reduced to $n(n-1)(n-2)$ if negative cycles are known not to exist. This method is therefore as efficient as the best result known, that of Murchland [3].

It is similar to many proposed schemes in that entries a_{ij} in the matrix are replaced by $a_{ik} + a_{kj}$ if the latter sum is smaller for some choice of k. After replacement the new matrix is operated upon in the same way until no improvement can be found. The various methods differ only in the rules for scanning the various (i, j) and k. In order to keep track of the routes as well as their values, it is also necessary to record for each (i,j) either the first arc of the minimum route from i to j or the last arc. With this information it is easy to generate all the arcs along the route. Aside from the efficiency, the second advantage of the method is the simplicity of the proof of its validity.

Assume for nodes $1, 2, \ldots, k-1$ that optimal distances $\bar{a}_{ij}$ are given, we wish to determine optimal distances a^*_{ij} for nodes $1, 2, \ldots, k$. We shall show that

For $\ell = 1, \ldots, (k-1)$

(1) $$a^*_{k\ell} = \min_j (a_{kj} + \bar{a}_{j\ell}) \qquad j = 1, 2, \ldots, k-1$$

(2) $$a^*_{\ell k} = \min_j (a_{jk} + \bar{a}_{\ell j})$$

(3) $$a^*_{kk} = \min_j [0, a^*_{kj} + a^*_{jk}]$$

For $(i = 1, \ldots, k-1)$ and $(j = 1, \ldots, k-1)$

(4) $$a^*_{ij} = \text{Min}[\bar{a}_{ij}, a^*_{ik} + a^*_{kj}]$$

The inductive procedure begins with $\bar{a}_{11} = 0$ and stops if at any time a diagonal value $a^*_{ii} < 0$ appears in which case a negative

cycle has been obtained; or if step $k = n$ has been completed.

<u>Proof</u>: (1) states that a minimum route from k to ℓ <u>starts</u> with some arc a_{kj} followed by a minimum route from j to ℓ that does not go through k . Hence the minimum of these alternative routes is the one desired.

Formula (2) is the same idea except the alternative routes are defined by the last arc a_{jk} of the route and the best route from ℓ to j that does not go through k .

Formula (3) states that either $a_{kk} = 0$ is the best route from k to k or there is a negative cycle consisting of going along some best route from k to i and then i back to k .

Formula (4) states that either the best route from (i, j) does not go through k (and has value $\bar{a}_{ij}$) or does go through k (and has value $a^*_{ik} + a^*_{kj}$).

The count on additions is

$$C = \sum_{k+1}^{n} [(k-1)(k-2) + (k-1)(k-2) + (k-1) + (k-1)^2]$$

where the four terms are the count of (1), (2), (3), (4) respectively. Note that we omitted from the count (for example) the addition $a_{k\ell} + \bar{a}_{\ell\ell}$ because $\bar{a}_{\ell\ell}$ is known to be zero. In the case that negative cycles are known not to exist, the third term may be dropped and the last term reduced by (k-1) since the diagonal $a^*_{ii} = 0$. In the latter case, the count is

$C = n(n-1)(n-2)$ additions and an equal number of comparisons.

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An inductive procedure on nodes is given that requires n(n-1)(n-2) comparison - addition operations to determine minimum routes between all nodes of a directed network. Arc distances may be negative. If negative cycles exist, however, termination will occur when one such is found.

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